Indian Statistical Institute
M.Math. II Year
First Semester Exam 2006-07
Advanced Probability

Time: 3 hrs | Date:27-11-06 | Max. Marks: 90
Instructor: B Rajeev

1. (a) Suppose (X_n) is an independent sequence of random variables. For $c > 0$ let $X_n^c := X_n I_{(|X_n| \leq c)}$. Suppose that $\sum P(|X_n| > c), \sum EX_n^c, \sum \mathrm{Var}(X_n^c)$ are all convergent. Show that $\sum X_n$ converges almost surely. Hint : If $EX_n = 0$ and $\sum \mathrm{Var}(X_n) < \infty$ then $\sum X_n$ converges a.s. (10)

   (b) By considering the distributions of $\frac{X_1 + \cdots + X_n}{n}$ where the (X_n) are appropriate i.i.d's with $P(X_n = \pm 1) = 1$, show that a sequence of distribution functions converging weakly to a limit distribution need not converge to the limit distribution at the points of discontinuity of the limit distribution. (10)

   (c) Let $h : \mathbb{R} \to \mathbb{R}$ be measurable and let $D_h = \{x : h$ is not continuous at $x\}$. If $\mu_n \Rightarrow \mu$ and $\mu(D_h) = 0$ then show that $\mu_n \circ h^{-1} \Rightarrow \mu \circ h^{-1}$. (6)

2. Let for each $n \geq 1$,$(X_{nk})_{k=1}^{r_n}$ be a finite sequence of independent random variables with $EX_{nk} = 0$ and $\sigma_{nk}^2 := EX_{nk}^2 < \infty$. Let $S_n = X_{n1} + \cdots + X_{nr_n}$ and $s_n^2 = Var(S_n)$. Suppose that Lyapunov's condition holds viz. for some $\delta > 0$
$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{k=1}^{r_n} E|X_{nk}|^{2+\delta} = 0$$
Show that $\frac{S_n}{\sqrt{s_n}} \Rightarrow N(0, 1)$. (6)

3. Let $\{\mu_i : i \geq 1\}$ be sequence of probability measures on $\mathbb{R}$. Show that there exists a stochastic process (X_n) on some probability space such that for each n, X_n has distribution μ_n. (11)

4. (a) Let F be a continuous distribution function on $\mathbb{R}$. Show that $F = F_{ac} + F_s$, where F_{ac} is absolutely continuous and F_s is singular. Show that the decomposition is unique. (10)

(b) Let X be an integrable random variable on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub σ-field. Show that the conditional expectation $E[X|\mathcal{G}]$ exists and is unique upto null sets. (10)

(c) Let X be a $\mathbb{R}^j$ valued random variable and Y be a $\mathbb{R}^k$ valued random variable on a probability space $(\Omega, \mathcal{F}, P)$. Suppose $\mathcal{G}$ is a sub σ-field of $\mathcal{F}$. Suppose that X is $\mathcal{G}$ measurable and Y is independent of the sigma field $\mathcal{G}$. Then show that

$$P((X, Y) \in H|\mathcal{G}) = f(X)$$

where $f(x) = P((x, Y) \in H)$. (10)

5. (a) Let X be a random variable on $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$. Let $\phi : I\!\!R \to I\!\!R$ be a measurable function such that $\phi(X)$ is integrable. Show that

$$E[\phi(X)|\mathcal{G}](\omega) = \int_{I\!\!R} \phi(x)\mu(dx, \omega) \quad \text{a.s}$$

where $\mu(dx, \omega)$is the conditional distribution of X given $\mathcal{G}$. (12).

(b) Let $(X_n, \mathcal{F}_n)_{n \geq 0}$ be a martingale. Let $(W_n)_{n \geq 1}$ be a sequence of non constant , bounded random variables. State sufficient conditions on the sequence (W_n) so that $(Y_n, \mathcal{F}_n)$ is a martingale, where

$$Y_n := X_0 + W_1 \Delta X_1 + \cdots + W_n \Delta X_n, \quad n \geq 0.$$

Prove your result. (5)